Game Theory. Examples of problems.

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1. *A Duopoly model.* Two firms, labeled 1 and 2, produce a product in quantities Q_1 and Q_2 respectively. The market price depends only on the total supply $Q = Q_1 + Q_2$:

$$P(Q) = \begin{cases} 5 - Q & \text{if} \quad Q < 5 \\ 0 & \text{if} \quad Q \geq 5. \end{cases}$$

The production costs are quadratic: $C(Q_i) = cQ_i^2$, where c is a positive constant. Each firm is aiming to maximise its profit.

(i) Suppose the firms choose the quantities Q_1, Q_2 independently (Cournot type model). Find the Nash equilibrium for this game.

(ii) Suppose the two firms collude by agreeing to produce the amount $Q = Q_1 = Q_2$ and that they have some way of enforcing this agreement (a Cartel model). What is the optimal choice $Q^\star$ for Q?

(iii) Choosing $c = 1$ for simplicity, confirm that under the Cartel agreement of (ii) the firms get higher profits than playing the Nash strategies obtained in (i).

2. *Cournot model for arbitrary number of firms.* N firms produce the quantities Q_1, Q_2,..., Q_m of some product on the same market. The market price is $P(Q) = P_0(1-Q/Q_0)$, where $Q = Q_1 + ... + Q_m$, and the profit of firm i is given by

$$\Pi_i(Q_1, ..., Q_m) = (P(Q) - c)Q_i,$$

$i = 1, ..., m$, where a positive constant $c < P_0$ denotes the marginal costs of production. Find symmetric Nash equilibria $Q_1^\star = ... = Q_m^\star = Q^\star$.

3. Consider a game given by the table

		C	
		1	2
R	1	a,b	c,d
	2	a,f	c,h

with arbitrary a, b, c, d, f, h. Let $p^\star = (h - f)/(h - f + b - d)$. Show that

(i) if $h > f$ and $b > d$, the Nash equilibria are $((p, 1 - p), (1, 0))$ for $p > p^\star$, $((p, 1 - p), (0, 1))$ for $p < p^\star$ and $((p^\star, 1 - p^\star), (q, 1 - q))$ for all q;

(ii) if $h < f$ and $b < d$, the Nash equilibria are $((p, 1 - p), (0, 1))$ for $p > p^\star$, $((p, 1 - p), (1, 0))$ for $p < p^\star$ and $((p^\star, 1 - p^\star)), (q, 1 - q)$ for all q;

(iii) if $h > f$, $b < d$ and $h - f > (d-b)/2$, the Nash equilibria are $((p, 1-p), (1, 0))$ for all p.

4. Show that any pair of strategies constitutes a Nash equilibrium for a game given by the table

C

R		1	2
	1	a,b	c,b
	2	a,f	c,f

Table 9.6

with arbitrary numbers a, b, c, f.

5. Find all symmetric Nash equilibria and ESS for a Hawk-Dove game of the form

	hawk	dove
hawk	0	V
dove	0	D

Table 9.7

for arbitrary numbers D and V.

6. Find all symmetric Nash equilibria and ESS for a pure coordination game with the table

C

R		1	2	3
	1	1,1	-1,-2	-1,-3
	2	-2,-1	2,2	-1,-2
	3	-3,-1	-3,-2	3,3

and write down the corresponding RD equations. Find the fixed points and check which of them are stable.

7. Consider a Hawk-Dove (or Lion-Lamb) type game G with payoff table

	A	B
A	-1,-1	5,0
B	0,5	3,3

Table 9.8

(i) Show that there is only one symmetric Nash equilibrium $\sigma^\star = (p^\star, 1-p^\star)$, find this equilibrium and prove that it defines ESS.

(ii) Write down the standard Replicator Dynamics (RD) equation for the proportion x of the players of the game G using action A with probability 1, identify the fixed points of this RD equation and show which of them are asymptotically stable and which are not.

(iii) Consider a game in which the game G is repeated an infinite number of times and payoffs are discounted by a factor δ $(0 < \delta < 1)$. Assume the players are limited to selecting strategies from the following 3 options:

(a) σ_A: play A in every stage game,

(b) σ_B: play B in every stage game,

(c) the "trigger strategy" σ_T: begin by playing B and continue to play B until your opponent plays A; once your opponent has played A, play A forever afterwards.

Find out whether $[\sigma_A, \sigma_A]$ and/or $[\sigma_B, \sigma_B]$ represent a Nash equilibrium, find a condition on δ such that $[\sigma_T, \sigma_T]$ is a Nash equilibrium, and find out whether $[\sigma_T, \sigma_T]$ is ESS.

(iv) Consider a modification of the game G, where the first player chooses his/her strategy and then advises his/her opponent about this choice, so that the second player makes his/her move depending on the choice of the first player. Draw the game tree (or extensive form) for this game and find a solution using the method of backward induction. Give the strategic form of this game and find all the pure Nash equilibria. Identify the subgames of this game and show that the solution found by backward induction is subgame perfect.

8. *Sherlock Holmes versus professor Moriarty.* This is an example from Conan Doyle's story "The final Problem" that was subjected to game theoretical analysis by von Neumann and Morgenstern. Holmes is going from London to Dover and then to the Continent in order to escape Moriarty who is going to kill Holmes as an act of revenge. When his train pulls out, Holmes observes Moriarty on the platform and realized that Moriarty might secure a special train to overtake him (reaching Dover earlier). So Holmes can either proceed to Dover or leave train at Canterbury, the only intermediate station. Moriarty is intelligent enough to visualize this possibility and consequently is faced with the same choice. Both leave the trains independently, and if they turn out to be on the same platform, Holmes would be almost certainly killed by his adversary. Assuming that Holmes's chances of survival are 100 per cent, if he escapes via Dover and 50 cent if he escapes via Canterbury (as in the latter case pursuit continues), the game can be represented by the following table

		Moriarty	
		Canterbury	Dover
Holmes	Canterbury	0	50
	Dover	100	0

Table 9.9

where Holmes's payoffs are shown, Moriarty's payoffs being the negatives of these.

Find the minimax strategies of Holmes and Moriarty and hence the value of the game.

9. *Game Morra.* This is a two-player symmetric zero-sum Italian game that is played by the following rules. Two players simultaneously extend one, two or three fingers and at the same time call out a number between one and three. The number called out is a guess of the number of fingers shown by the opponent. If only one player guesses correctly, then the loser pays the winner the amount corresponding to the total number of fingers displayed, otherwise no payoff is due. The matrix of the game is

C

R

	1-1	1-2	1-3	2-1	2-2	2-3	3-1	3-2	3-3
1-1	0	2	2	-3	0	0	-4	0	0
1-2	-2	0	0	0	3	3	-4	0	0
1-3	-2	0	0	-3	0	0	0	4	4
2-1	3	0	3	0	-4	0	0	-5	0
2-2	0	-3	0	4	0	4	0	-5	0
2-3	0	-3	0	0	-4	0	5	0	5
3-1	4	4	0	0	0	-5	0	0	-6
3-2	0	0	-4	5	5	0	0	0	-6
3-3	0	0	-4	0	0	-5	6	6	0

Table 9.10

Find the solutions (minimax strategies) of this game.

10. Suppose that a two players two actions zero sum game is given by a matrix $A = (a_{ij})$, $i, j = 1, 2$. (i) Show that if there do not exist pure strategy Nash equilibria (or saddle points), then either

$$1) \quad \max(a_{11}, a_{22}) \le \min(a_{12}, a_{21}), \quad a_{11} + a_{22} < a_{12} + a_{21},$$

or

$$2) \quad \max(a_{12}, a_{21}) \le \min(a_{11}, a_{22}), \quad a_{11} + a_{22} > a_{12} + a_{21}.$$

(ii) Show that in both cases the value of the game is $\det(A)/t(A)$, where $\det(A)$ is the determinant of A and $t(A) = a_{11} + a_{22} - a_{12} - a_{21}$, and

$$\sigma = \left(\frac{a_{22} - a_{21}}{t}, \frac{a_{11} - a_{12}}{t} \right), \quad \eta = \left(\frac{a_{22} - a_{12}}{t}, \frac{a_{11} - a_{21}}{t} \right)$$

is a pair of minimax strategies.